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LETTER TO THE EDITOR

Extended irreversible thermodynamics and runaway electrons in plasmas

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Abstract. The stability of the steady state of a completely ionised plasma under an external electric field is analysed from extended irreversible thermodynamics. Fourth-order terms in the electric current are included in the generalised entropy and fluctuation theory is used to obtain numerical values for the coefficients. The final result is in good agreement with evaluations from a Fokker–Planck equation in plasma theory.

Extended irreversible thermodynamics [1–6] is a generalisation of non-equilibrium thermodynamics which has received increasing attention in the past few years. As a special feature, it includes dissipative fluxes amongst the independent variables of a generalised entropy. Unlike the classical theory [7, 8], which is recovered in the limit of vanishing relaxation time of the fluxes, it leads to relaxational evolution equations for the fluxes which are useful for the description of high-frequency phenomena [9] and to generalised equations of state with non-equilibrium corrections [10]. The connection between the dynamics of the fluxes and non-equilibrium equations of state is one of the main open topics in recent non-equilibrium thermodynamic theories [11–13], though it does not even arise in the usual non-equilibrium thermodynamic theory [7, 8]. The latter one is based on the local-equilibrium hypothesis, so that it assumes *a priori* that the equations of state have in non-equilibrium the same functional form as in equilibrium, but with a local meaning, and that the fluxes decay instantaneously to the values given by the classical transport laws of Fourier, Ohm, Fick, Navier–Stokes, etc.

Here, we study an instability of the electric current in a classical plasma under an electric field, as predicted by extended irreversible thermodynamics. The stability analysis is a crucial point in the understanding of the generalised equations of state, so that it deserves a careful analysis either from numerical and experimental perspectives. In the situation analysed here, we compare our thermodynamic predictions with the results obtained from a Fokker–Planck equation in plasma theory [14].

We take as independent variables of the entropy the classical ones (internal energy per unit mass u, specific volume per unit mass v, electron number density n_e) plus the electric current density $\boldsymbol{J}$ [15, 16]. The generalised entropy per unit mass has the form, up to fourth order in the electric current,

$$s(u, v, n_e, \boldsymbol{J}) = s_{eq}(u, v, n_e) - \alpha \boldsymbol{J}\cdot\boldsymbol{J} - \beta(\boldsymbol{J}\cdot\boldsymbol{J})^2 \tag{1}$$

with $s_{eq}(u, v, n_e)$ the local-equilibrium entropy. The coefficients α and β may be calculated from fluctuation theory. Indeed, according to Callen's theory of fluctuations

[17], one has for the second and the fourth moments of fluctuations of $\boldsymbol{J}$

$$\langle(\delta J_x)^2\rangle = k/2\alpha \tag{2}$$

$$\langle(\delta J_x)^4\rangle = -(3\beta k^3/2\alpha^4) + (3k^2/4\alpha^2). \tag{3}$$

These relations allow one to compute the coefficients α and β from $\langle(\delta J_x)^2\rangle$ and $\langle(\delta J_x)^4\rangle$, the angular brackets meaning equilibrium averages.

In the presence of an electric field, one is led for the second differential of the entropy to

$$(\partial^2 s/\partial \boldsymbol{J}^2) = -2(\alpha + 6\beta \boldsymbol{J}_0^2) \tag{4}$$

with $\boldsymbol{J}_0 = \sigma \boldsymbol{E}$, the mean value of $\boldsymbol{J}$ in the steady state, σ being the electrical conductivity of the plasma. The thermodynamic stability theory [17, 18] requires, amongst other conditions, that $(\partial^2 s/\partial \boldsymbol{J}^2)$ must be negative. The negative character of the second differential of the entropy is also related to the hyperbolicity of the evolution equations of the system [19] so that it may have also a dynamical meaning.

From (4) it is seen that the second derivative with respect to J becomes positive and the corresponding steady state is no longer stable when the electric field E becomes higher than the critical value

$$E_{\text{crit}} = (1/\sigma)(-\alpha/6\beta)^{1/2}. \tag{5}$$

We now turn to the explicit calculation of α and β from equilibrium statistical mechanics. We have for the moments of the fluctuations of the fluxes

$$\langle(\delta J_x)^2\rangle = \int \mathrm{d}\boldsymbol{c} \int \mathrm{d}\boldsymbol{c}'\, e^2 c_x c_x' \langle \delta f(\boldsymbol{c}) \delta f(\boldsymbol{c}')\rangle \tag{6}$$

and

$$\langle(\delta J_x)^4\rangle = \int \mathrm{d}\boldsymbol{c} \int \mathrm{d}\boldsymbol{c}' \int \mathrm{d}\boldsymbol{c}'' \int \mathrm{d}\boldsymbol{c}'''\, e^4 c_x c_x' c_x'' c_x''' \langle \delta f(\boldsymbol{c}) \delta f(\boldsymbol{c}') \delta f(\boldsymbol{c}'') \delta f(\boldsymbol{c}''')\rangle \tag{7}$$

where we have taken into account that

$$\boldsymbol{J} = \int e\boldsymbol{c} f(\boldsymbol{c})\, \mathrm{d}\boldsymbol{c}. \tag{8}$$

Here, $f(\boldsymbol{c})$ is the velocity distribution function, $\boldsymbol{c}$ is the relative particle velocity with respect to the barycentric motion of the system and e is the electron charge.

The second and fourth moments of the fluctuations in the velocity distribution function may be obtained from Callen's theory [17] by starting from the Boltzmann definition of the entropy

$$s = -kv \int f(\boldsymbol{c}) \ln f(\boldsymbol{c})\, \mathrm{d}\boldsymbol{c} \tag{9}$$

where s and v are entropy and volume per unit mass, respectively. To apply Callen's formalism, we define the thermodynamic conjugate of $f(\boldsymbol{c})$ as

$$\begin{aligned} F(\boldsymbol{c}, \boldsymbol{c}') &= -\partial[kvf(\boldsymbol{c}') \ln f(\boldsymbol{c}')]/\partial f(\boldsymbol{c}) \\ &= -kv[1 + \ln f(\boldsymbol{c})]\delta(\boldsymbol{c} - \boldsymbol{c}') \end{aligned} \tag{10}$$

where $\delta(\boldsymbol{c}-\boldsymbol{c}')$ is Dirac's delta. Callen's expressions for the second and fourth moments of fluctuations lead then to

$$\langle \delta f(\boldsymbol{c})\delta f(\boldsymbol{c}')\rangle = (1/v)f(\boldsymbol{c})\delta(\boldsymbol{c}-\boldsymbol{c}') \tag{11}$$

and

$$\begin{aligned}\langle \delta f(\boldsymbol{c})\delta f(\boldsymbol{c}')\delta f(\boldsymbol{c}'')\delta f(\boldsymbol{c}''')\rangle \\ = (1/v^3)f(\boldsymbol{c})\delta(\boldsymbol{c}-\boldsymbol{c}')\delta(\boldsymbol{c}'-\boldsymbol{c}'')\delta(\boldsymbol{c}''-\boldsymbol{c}''') \\ +(1/v^2)f(\boldsymbol{c})f(\boldsymbol{c}')\delta(\boldsymbol{c}-\boldsymbol{c}''')\delta(\boldsymbol{c}'-\boldsymbol{c}'') \\ +(1/v^2)f(\boldsymbol{c})f(\boldsymbol{c}'')\delta(\boldsymbol{c}''-\boldsymbol{c}''')\delta(\boldsymbol{c}-\boldsymbol{c}') \\ +(1/v^2)f(\boldsymbol{c})f(\boldsymbol{c}''')\delta(\boldsymbol{c}-\boldsymbol{c}'')\delta(\boldsymbol{c}'-\boldsymbol{c}''').\end{aligned} \tag{12}$$

When (11) and (12) are introduced into the macroscopic relations (2) and (3), and taking for $f(\boldsymbol{c})$ the Maxwell-Boltzmann equilibrium distribution function, one obtains for the parameters α and β

$$\alpha = k(2e^2n_e^2kT)^{-1} \qquad \beta = -km(8e^4n_e^4k^2T^2)^{-1}. \tag{13}$$

From this result we may calculate the critical value of the electric field as given by (5), which turns out to be

$$E_{\text{crit}} = (2/3)^{1/2}(ne/\sigma)(kT/m)^{1/2}. \tag{14}$$

It remains to look for some independent corroboration of this result. We turn our attention to plasma physics [14]. Here, the analysis of a fully ionised classical plasma under the action of an external electric field may be carried out by starting from the corresponding Fokker-Planck equation. If the electric field is small enough, a steady state may be reached in which the acceleration of the particles by E is balanced by the collisional drag. If the field is too strong, however, a steady state may not be reached, but the electrons acquire increasing speed and run away. The critical electric field beyond which there is no steady state relation between collisions and the driving field as calculated from the Fokker-Planck equation turns out to be

$$E_{\text{crit}} = 0.99(2/3)^{1/2}(ne/\sigma)(kT/m)^{1/2}. \tag{15}$$

Rather than rely on the good agreement between the macroscopic result (14) and the kinetic evaluation (15), we insist on the posible physical significance of the fourth-order terms in the entropy. Such terms have been introduced here for the first time in extended irreversible thermodynamics, which in previous versions was limited to second-order terms. Furthermore, fluctuation theory reveals itself as a useful tool for the computation of the coefficients appearing in the entropy, which are essential for the evaluation of our results.

Note finally that the increment in velocity produced by an electric field of the order of (14) between two successive collisions is of the order of the variance of the velocity distribution function. After several collisions, this may produce considerable separations with respect to the Maxwellian distribution, leading to instabilities.

As has been seen in this letter, extended thermodynamics does not merely lead to only very minor corrections to the standard thermodynamic theory, but may be useful for the description of quite relevant phenomena.

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